Phase change in a diffracted wave: a Cornu spiral perspective

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A simple evaluation of the phase change in a diffracted wave, in terms of the Cornu spiral, is presented to complement the well-known intensity change, which is routinely obtained for this elegant graphical construction of the Fresnel integrals. This is, to the best of our knowledge, the first presentation of this evaluation. It is shown that the phase of a wave diffracted by a slit is equal to the slope of the line tangent to the Cornu spiral, shifted by $\pi/4$. © 2010 Optical Society of America

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The Cornu spiral—named after the French physicist Marie Alfred Cornu, and also known as the Euler spiral or clothoid—has been used in a number of applications. For example, this spiral is an ideal curve for keeping a constant speed in curves in railroads and highways [1], because its curvature changes linearly with the curve length. In fact, Castaño [2] elegantly showed that, if one imposes to a curve the condition that its curvature is proportional to the arc length, then the curve is a Cornu spiral. Other applications of this particular spiral go from roller-coaster loop shapes [3] to mobile robot trajectories [4]. In the field of diffraction physics, the Cornu spiral is used to graphically estimate the intensity of diffracted waves when the Fresnel approximation is applied [5]. Here we show for the first time, to the best of our knowledge, that the phase of a diffracted wave can also be very simply related to a geometrical property of a clothoid and to its arc length.

Let us consider an infinite slit on the plane Y–Z, of aperture size $2z$ (see Fig. 1). The plane of the slit is perpendicular to the line separating the source S and the point where the diffracted wave is evaluated at P. The origin O of the Y–Z Cartesian coordinate system is aligned with (SP). Distances from S to O and from O to P are ρ_0 and r_0, respectively. The wave at P is obtained, as usual, by the summation of all the secondary waves arising from points $A(y,z)$ inside the slit. The distance between S and A is designated by ρ, and that between A and P by r. Using the Fresnel approximation to the Kirchhoff diffraction integral, the complex amplitude at P is given by [5,6]

$$\Psi_P = -i\frac{E_0}{\lambda r_0\rho_0}\int_{-\infty}^{\infty}\int_{-z}^{z}\mathrm{d}z\mathrm{d}y \times \exp\left[ik\left(\rho_0 + r_0 + (y^2+z^2)\frac{\rho_0+r_0}{2\rho_0 r_0}\right)\right], \quad (1)$$

where λ is the wavelength and E_0 is the amplitude of the wave at S. Introducing the dimensionless variables

$$u = y\left[\frac{2(\rho_0+r_0)}{\lambda\rho_0 r_0}\right]^{1/2}, \qquad v = z\left[\frac{2(\rho_0+r_0)}{\lambda\rho_0 r_0}\right]^{1/2} \quad (2)$$

and by performing the corresponding change of variables in Eq. (1), one obtains

$$\Psi_P = -i\frac{E_0}{2(r_0+\rho_0)}\exp(ik(r_0+\rho_0))\int_{-\infty}^{\infty}\int_{-v}^{v}\mathrm{d}u\mathrm{d}v \times \exp\left[i\frac{\pi}{2}(u^2+v^2)\right], \quad (3)$$

which can be written as

$$\Psi_P = B(R+iI), \quad (4)$$

where

$$B = -i\frac{E_0}{2(r_0+\rho_0)}\exp(ik(r_0+\rho_0)), \quad (5)$$

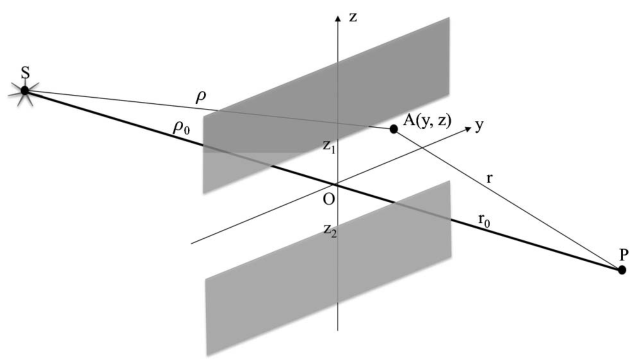

Fig. 1. Geometry of the diffraction by a slit.

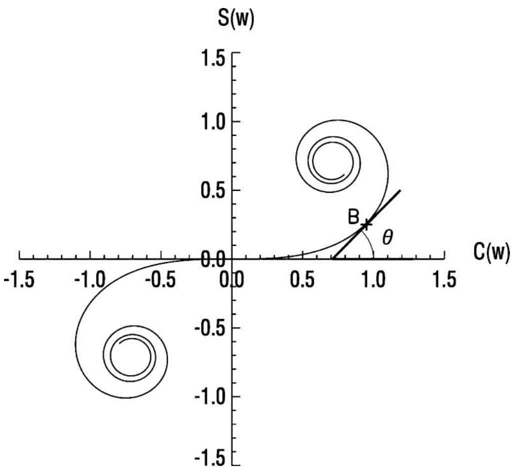

Fig. 2. Cornu spiral. The coordinates are given by Eqs. (8) and (9).

$$R = \int_{-\infty}^{\infty} \int_{-v}^{v} \mathrm{d}u\mathrm{d}v \cos\left[\frac{\pi}{2}(u^2 + v^2)\right], \tag{6}$$

$$I = \int_{-\infty}^{\infty} \int_{-v}^{v} \mathrm{d}u\mathrm{d}v \sin\left[\frac{\pi}{2}(u^2 + v^2)\right]. \tag{7}$$

The Fresnel integrals are defined as

$$\mathcal{C}(w) = \int_{0}^{w} \mathrm{d}\zeta \cos\left(\frac{\pi}{2}\zeta^2\right), \tag{8}$$

$$\mathcal{S}(w) = \int_{0}^{w} \mathrm{d}\zeta \sin\left(\frac{\pi}{2}\zeta^2\right). \tag{9}$$

Considering that $\mathcal{C}(\infty) = \mathcal{S}(\infty) = 1/2$ and that both functions are odd, Eqs. (6) and (7) can be written in terms of the Fresnel integrals as

$$R = 2(\mathcal{C}(w) - \mathcal{S}(w)), \tag{10}$$

$$I = 2(\mathcal{C}(w) + \mathcal{S}(w)), \tag{11}$$

where w takes the value

$$w = \frac{2(\rho_0 + r_0)}{\lambda \rho_0 r_0} z. \tag{12}$$

An elegant graphical representation of the Fresnel integrals is the Cornu spiral, where $\mathcal{C}$ and $\mathcal{S}$ are regarded as rectangular Cartesian coordinates of a point B. As w takes all the possible values, the point B describes the Cornu spiral (see Fig. 2). It is well known in the literature [5] that the intensity of the wave at P is proportional to the distance from the origin of the Cornu spiral plot to the point on the spiral associated to the value of w given in Eq. (12). For the case of the phase, to our knowledge, there is no known relation between the phase of the wave at P and the Cornu spiral. Here we shall show that they are related by a very simple equation. From Eqs. (4) and (5), the phase of the complex amplitude at P is given by

$$k(r_0 + \rho_0) + \phi, \tag{13}$$

where ϕ is defined by $\tan(\phi) = I/R$. Substituting into the former equation the expressions of R and I given by Eqs. (10) and (11), we obtain

$$\tan(\phi) = \frac{\mathcal{C}(w) + \mathcal{S}(w)}{\mathcal{C}(w) - \mathcal{S}(w)} = \frac{1 + \mathcal{S}(w)/\mathcal{C}(w)}{1 - \mathcal{S}(w)/\mathcal{C}(w)} = \frac{1 + \tan(\theta)}{1 - \tan(\theta)} = \tan\left(\theta + \frac{\pi}{4}\right), \tag{14}$$

where θ is the angle between the axis representing $\mathcal{C}$ and the tangent to the spiral at point B (see Fig. 2), and the last equality is a known trigonometric identity. Furthermore, it is known [5] that $\theta = (\pi/2)w^2$. Thus, we have the simple relation

$$\phi = \frac{\pi}{2}w^2 + \frac{\pi}{4} \text{modulo } \pi. \tag{15}$$

The simple analytic expression of the diffracted phase given by Eqs. (12) and (15) permits a very fast evaluation of the phase using minimal computational resources. Fast algorithms for the numerical calculation of Fresnel patterns have been studied by a number of authors [7–15]. The elegant and simple result reported here can be very useful for the development of such algorithms, which can be applied to different fields of optics, such as the calculation of defocused Fresnel images [16], the propagated fields inside a theoretical eye [17–19], and the computation of the average phase of a beam over a detector in the near field for the Space Interferometry Mission [20].

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